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Dynamical O(4) symmetry in the asymptotic field of the Prasad–Sommerfield monopole

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Abstract. The exact solution is presented for the classical and quantum mechanical problem of a spinless, isospin-carrying test particle interacting with a singular monopole of the same large-distance asymptotic behaviour as the Prasad–Sommerfield 1-monopole. It is demonstrated that this problem has a dynamical O(4) symmetry. The classical mechanical equation of motion is traced back to a non-relativistic Coulomb problem. The long-range Higgs field results in monopole-test-particle bound states. The bound state spectrum is derived on a purely group theoretic basis using dynamical symmetry.

1. Introduction

The non-relativistic Coulomb problem (e.g. Bander and Itzykson 1966), the interaction of two dually charged particles (Zwanziger 1968), the interaction of an electric point charge with the Dirac monopole (Jackiw 1980, Bacry 1981, Barut 1981, Golo 1982, Horváthy 1983) all have certain dynamical symmetries. It was noticed by several authors (Jackiw 1980, Golo 1982, Barut and Bracken 1983) that the Wu–Yang (1968) monopole and/or the 't Hooft–Polyakov monopole ('t Hooft 1974, Polyakov 1974) could also have a dynamical invariance.

The following one-parameter family of solutions (Protogenov 1977) in the standard SU(2) Yang–Mills–Higgs model in the Prasad–Sommerfield limit:

$$
\begin{aligned}
&A_0^a = 0, \qquad \phi^a = \frac{x^a}{r^2} H(r), \qquad (a = 1, 2, 3),\\
&A_j^a = \varepsilon_{ajk} \frac{x^k}{r^2} [1 - K(r)], \qquad (j, k = 1, 2, 3), \\
&H(r) = r \coth(r + r_0) - 1, \qquad K(r) = \frac{r}{\sinh(r + r_0)}
\end{aligned}
\tag{1.1}
$$

reduces to the Prasad and Sommerfield (1975) 1-monopole for $r_0 = 0$, and for other values of the parameter r_0 it describes a singular monopole of the same large-distance asymptotic behaviour.

In this paper we present the exact solution for the classical and quantum mechanical problem of a spinless, isospin-carrying test particle in the background field (1.1) with $r_0 = \infty$. We demonstrate that this problem has a dynamical O(4) symmetry quite analogous to that of the non-relativistic Coulomb problem. As a trivial example of the dimensional reduction procedure (Forgács and Manton 1980, Harnad *et al* 1980a) the adjoint Higgs field ϕ^a can be regarded as the extra spacelike component A_4^a of a Yang-Mills potential A_μ^a ($\mu=0,\ldots,4$) over a five-dimensional flat spacetime. Our method is to investigate the mechanics in the five-dimensional pure gauge field corresponding to (1.1) (Fehér 1984; see Shnider and Sternberg 1983 also).

In § 2 we present the test particle's classical mechanics in a form adapted to the developments in §§ 3 and 4.

The third part of the paper is devoted to an analysis of the classical motion in the presence of the '$r_0=\infty$ monopole' and its symmetry algebra. The spatial motion takes place on the surface of a rotation cone whose axis is the total angular momentum vector. We shall apply the 'trick' of Boulware *et al* (1976), i.e. to bend down the cone in question to the plane perpendicular to its axis. The clue to the exact solution is that this transformation leads to an effective, non-relativistic Coulomb problem having a well known O(4) symmetry.

In the fourth part we show how the dynamical symmetry does work in the quantum mechanical version of the problem. There exists a rich bound state spectrum which follows from the dynamical O(4) symmetry. From a physical point of view the bound states are produced by the attractive Higgs field coupling. The Higgs field is of zero mass in the Prasad-Sommerfield limit and therefore cannot be neglected as was done in Schechter (1976) and Marciano and Muzinich (1983).

Din and Roy (1983) solved the Dirac equation for an isospinor fermion in the $r_0=\infty$ monopole's field. Most likely the five-dimensional version of this problem is also completely soluble using group theoretic arguments only. The presence of the dynamical symmetry was conjectured in Fehér (1985) from an explicit solution of the Klein-Gordon equation.

2. Preliminaries on classical mechanics

Our aim is to investigate the mechanics of a pointlike, spinless, isospin-carrying test particle in the background field (1.1) with $r_0=\infty$. Throughout the paper we work in the rest frame of the monopole and 'before' the pure Yang-Mills $\rightarrow$ Yang-Mills-Higgs dimensional reduction.

Let M denote the five-dimensional flat spacetime without the worldline of the monopole. In cartesian coordinates x^μ ($\mu=0,\ldots,4$) the metric tensor is given by $g_{\mu\nu}=\operatorname{diag}(-1,1,1,1,1)$ and $g_{jk}x^jx^k=r^2\neq 0$, (latin indices j, k, ... run as 1, 2, 3).

In $\mathscr{G}$, the Lie algebra of SU(2), we choose a basis T_a ($a=1,2,3$) satisfying $[T_a, T_b]=\varepsilon_{abc}T^c$. We identify $\mathscr{G}$ and its dual space $\mathscr{G}^*$ with the aid of the adjoint invariant metric $\langle,\rangle_{\mathscr{G}}$ on $\mathscr{G}$ defined in the fixed basis by $\langle T_a, T_b\rangle_{\mathscr{G}}=\delta_{ab}$. Let ξ^a ($a=1,2,3$) denote the coordinates in $\mathscr{G}$ with respect to the basis T_a. Lie algebra indices can be raised and lowered by δ_{ab}.

An SU(2) Yang-Mills field over M appears as a connection ω on a principal fibre bundle $P(M, \mathrm{SU}(2), \pi)$ (Kobayashi and Nomizu 1963, Bleecker 1981). In our case P is equivalent to the product bundle $M\times\mathrm{SU}(2)$. From now on we fix a global section σ of P and work in the corresponding trivialisation. A connection ω on P is defined by its pull-back $\sigma^*\omega=A_\mu^a\,\mathrm{d}x^\mu\otimes T_a$. Let A_μ^a be given by (1.1) with $r_0=\infty$ and applying the substitution $\phi^a\rightarrow A_4^a$ there.

The classical mechanics of an isospin-carrying test particle in a Yang-Mills field can be treated in several ways (Kerner 1968, Wong 1970, Sternberg 1977, Balachandran *et al* 1977, Weinstein 1978, Sniatycki 1979, Duval and Horváthy 1982). Now we present it for our special case in a form (Montgomery 1984) particularly convenient for the objectives of §§ 3 and 4.

Let $P^{\#}$ be the pull-back of P to T^*M, i.e. $P^{\#} = P \times T^*M|_{\mathrm{diag}(M\times M)}$. The phase space for our particle is the coadjoint bundle $E^{\#} = P^{\#} \times_{\mathrm{SU}(2)} \mathscr{G}^*$. The section σ defines trivialisations of $P^{\#}$ and $E^{\#}$ through the trivialisation of P. The coordinates x^μ induce canonical coordinates (x^μ, p_ν) in T^*M. Taking these into account and using the fixed basis T_a of $\mathscr{G} \simeq \mathscr{G}^*$ we introduce coordinates (x^μ, p_ν, ξ^a) in $E^{\#} \simeq T^*M \times \mathscr{G}^*$. Let us define the Poisson brackets of these coordinate functions as

$$\begin{aligned} &\{x^\mu, x^\nu\} = 0, \qquad \{x^\mu, \xi^a\} = 0, \qquad \{\xi^a, \xi^b\} = -\varepsilon^{abc}\xi_c, \\ &\{\xi_a, p_\mu\} = \varepsilon_{abc}A^b_\mu \xi^c, \qquad \{p_\mu, p_\nu\} = -F^a_{\mu\nu}\xi_a, \qquad \{p_\mu, x^\nu\} = \delta^\nu_\mu. \end{aligned} \tag{2.1}$$

Here

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^a{}_{bc}A^b_\mu A^c_\nu \tag{2.2}$$

is the field strength tensor. For arbitrary f, $h \in C^\infty(E^{\#})$ we take

$$\{f, h\} = \sum_{\alpha,\beta} \frac{\partial f}{\partial z^\alpha}\frac{\partial h}{\partial z^\beta}\{z^\alpha, z^\beta\} \tag{2.3}$$

where z^α, z^β run through the coordinates (x^μ, p_ν, ξ^a) independently. In this way we have obtained a Poisson bracket operating in $C^\infty(E^{\#}) \times C^\infty(E^{\#})$. All the usual identities hold for the bracket $\{\,,\}$. However from $\{f, h\} = 0\ \forall h \in C^\infty(E^{\#})$ it does not follow now that f is a constant. In fact, the particle's total isospin

$$W = \delta_{ab}\xi^a\xi^b \tag{2.4}$$

is a Casimir function of the Poisson manifold (Weinstein 1983) $(E^{\#}, \{\,,\})$, i.e. W has a vanishing bracket with any observable.

The Hamiltonian equations with the kinetic Hamiltonian

$$\mathscr{K} = -\tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu \tag{2.5}$$

give the Wong equations (Wong 1970) governing the motion of the test particle:

$$\mathrm{d}\xi_a/\mathrm{d}s = \{\xi_a, \mathscr{K}\} = -\varepsilon_{abc}A^b_\mu p^\mu \xi^c, \tag{2.6a}$$

$$\mathrm{d}x^\mu/\mathrm{d}s = \{x^\mu, \mathscr{K}\} = p^\mu, \tag{2.6b}$$

$$\mathrm{d}p_\mu/\mathrm{d}s = \{p_\mu, \mathscr{K}\} = F^a_{\mu\nu}\xi_a p^\nu. \tag{2.6c}$$

In the first part of § 3 we shall describe conserved observables that are due to the symmetries of the monopole. Let us sketch the main ideas behind their derivation. For this we need the notion of the automorphism group Aut P of a principal fibre bundle $P(M, \mathrm{G}, \pi)$: Aut P consists of elements F from Diff P commuting with the right translation $R_g \in$ Diff P for any g from the structure group G. There exists a natural homomorphism η: Aut $P \to$ Diff M for which $\pi \circ F = \eta(F) \circ \pi$ is valid. Aut P acts on M by this homomorphism and therefore on T^*M as well. Furthermore, we regard Aut P acting on $P^{\#}$ and on $E^{\#}$ by means of its actions on P and on T^*M too.

If $F \in$ Aut P preserves the connection ω, i.e.

$$F^*\omega = \omega \tag{2.7}$$

holds, then its action on $E^{\#}$ preserves the Poisson bracket (locally it can be described in the general case as in (2.1)) because the definition of $\{\,,\}$ depends just on the connection. If in addition, $\eta(F) \in$ Diff M is an isometry of the spacetime metric then the kinetic Hamiltonian $\mathscr{H}$ and so the Wong equations also remain invariant with respect to the transformation F. Let the subgroup of Aut P consisting of elements having the above two properties be denoted by K_ω. K_ω is the usual invariance or symmetry group of the Yang-Mills field represented by ω (Forgács and Manton 1980, Harnad *et al* 1980b).

It is natural to call a vector field Y on P an infinitesimal symmetry (with respect to ω and the spacetime metric) if its flow $F_t \in \mathrm{Diff}\, P$ is in K_ω for sufficiently small t. From the above line of reasoning it is clear that an infinitesimal symmetry always provides us with a conserved quantity for the Wong equations (for details see Duval and Horváthy 1982). This fact will be of frequent use in the following.

3. The motion of the test particle and its symmetry algebra

Let us introduce an SU(2)-equivariant function $\tilde{\varphi}: P \to \mathscr{G}$ by

$$\begin{aligned} &\tilde{\varphi}(x, g) = \mathrm{Ad}_{g^{-1}}[\varphi(x)], \qquad (P \simeq M \times \mathrm{SU}(2)), \\ &\varphi^a(x) = x^a/r, \qquad \varphi(x) = \varphi^a(x) T_a. \end{aligned} \tag{3.1}$$

With the $r_0 = \infty$ monopole we have $(D\varphi)^a_\mu = \partial\varphi^a/\partial x^\mu + \varepsilon^a{}_{bc} A^b_\mu \varphi^c = 0$. From this and (2.6*a*)-(2.6*b*) it follows that

$$Q = \varphi^a \xi_a = \frac{1}{r} x^a \xi_a \tag{3.2}$$

is a conserved quantity. Choosing an arbitrary $y_0 \in P\tilde{\varphi}$ defines a U(1) sub-bundle P_0 of P by the equation

$$P_0 = \{y | y \in P, \tilde{\varphi}(y) = \tilde{\varphi}(y_0)\}. \tag{3.3}$$

The connection ω describing the monopole reduces to a connection ω_0 on P_0 because of $D\varphi = 0$ (Kobayashi and Nomizu 1963). As a matter of fact P_0 is the holonomy bundle of ω with base point y_0. So the $r_0 = \infty$ monopole is an embedded U(1), 'purely electromagnetic' solution in the five-dimensional SU(2) Yang-Mills model. The field strength tensor of ω_0 is given by

$$\mathscr{F}_{\mu\nu} = F^a_{\mu\nu}\varphi_a \tag{3.4}$$

independently on y_0. The motion of the test particle is determined by the five-dimensional Lorentz force law

$$\mathrm{d}x^\mu/\mathrm{d}s = p^\mu, \qquad \mathrm{d}p_\mu/\mathrm{d}s = Q\mathscr{F}_{\mu\nu}p^\nu \tag{3.5}$$

as it is easy to see from (2.6) and (1.1) with $r_0 = \infty$. This makes it plausible to identify the observable $Q \in C^\infty(E^\#)$ with the electric charge of the test particle. As a byproduct we now interpret the above as a non-trivial example of an interesting general result.

For any principal fibre bundle $P(M, G, \pi)$ there is a one-to-one correspondence between the G-equivariant functions from P to $\mathscr{G}$ and the adjoint Higgs fields, i.e. global sections of the adjoint bundle $E = P\times_G \mathscr{G}$. If a connection ω is given on P then this correspondence and the equation

$$\tilde{\varphi} = \omega(Y), \qquad \pi_*(Y) = 0 \tag{3.6}$$

lead to a one-to-one map from the covariantly constant Higgs field to the so-called infinitesimal internal symmetries (with respect to ω). An infinitesimal symmetry Y

with $\pi_*(Y) = 0$ is called 'internal' since its flow gives the identity transformation on the spacetime M. The Lie algebra of the infinitesimal internal symmetries of a connection ω is isomorphic to the centraliser of the holonomy algebra of ω in $\mathscr{G}$ (Fischer 1982, Bleecker 1984, Horváthy and Rawnsley 1984). If a test particle is coupled to the Yang-Mills field ω then a conserved gauge invariant charge $Q_Y \in C^\infty(E^\#)$ belongs to any infinitesimal internal symmetry Y. Q_Y is described by a formula analogous to (3.2) and so it is a gauge invariant component of the particle's $\mathscr{G}$-valued charge conserved only covariantly (see (2.6a)-(2.6b)). From the quoted result one gets that the subalgebra in $(C^\infty(E^\#), \{\,,\})$ which is formed by the conserved gauge invariant charges of the test particle is isomorphic to the centraliser of the holonomy algebra of ω in $\mathscr{G}$ (Horváthy and Rawnsley 1984). For the $r_0 = \infty$ monopole the holonomy algebra is one-dimensional and equal to its own centraliser. So Q in (3.2) is the only gauge invariant conserved charge of our test particle.

Turning back to our problem let $\bar{M}$ be the complete five-dimensional spacetime. First of all we are interested in the spatial motion of the test particle. The physical 3-space can be regarded as the plane $x^0 = 0$, $x^4 = 0$ in $\bar{M}$. In 3-vector notation (3.5) is

$$\begin{aligned} &\frac{\mathrm{d}p_0}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}p_4}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{Q}{r}, \\ &\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s} = \frac{Q}{r^3}[\boldsymbol{r} \times \boldsymbol{p} + p_4\boldsymbol{r}]. \end{aligned} \tag{3.7}$$

From the infinitesimal symmetry generated by the one-parameter family of symmetries of the monopole

$$F_t^4\{[(x^0, \boldsymbol{r}, x^4), g]\} = [(x^0, \boldsymbol{r}, x^4 + t), g] \tag{3.8}$$

one obtains the following constant of motion

$$Z = p_4 + Q(1 - 1/r), \qquad (Z \in C^\infty(E^\#)). \tag{3.9}$$

From (3.7) and (3.9) the spatial motion is governed by

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s} = Q\boldsymbol{r} \times \boldsymbol{p} - \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{r}}\left(\frac{Q(Z - Q)}{r} + \frac{1}{2}\frac{Q^2}{r^2}\right). \tag{3.10}$$

From the viewpoint of dimensional reduction (3.10) means that the Higgs field influences the test particle by means of long-range $1/r$ and $1/r^2$ potentials.

The mass m of our particle

$$m^2 = (p_0)^2 - g^{jk}p_jp_k \tag{3.11}$$

varies as a consequence of the Higgs field coupling. The positivity of m^2 is ensured by regarding only timelike worldlines in $\bar{M}$. We have the usual equality

$$p_0 = m\left[1 - \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}x^0}\right)^2\right]^{-1/2} \tag{3.12}$$

and p_0 is conserved because of the static and purely magnetic ($F^a_{0\mu} = 0$) character of the background field.

In the fixed trivialisation of P we can define the 'diagonal left action' of SU(2) on P by

$$F_g\{[(x^0, \boldsymbol{r}, x^4), g_0]\} = [(x^0, \eta(g)\boldsymbol{r}, x^4), gg_0], \qquad (\forall (x^0, \boldsymbol{r}, x^4) \in M, g, g_0 \in \mathrm{SU}(2)). \tag{3.13}$$

Here $\eta : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the covering homomorphism and $\eta(g)$ acts on the 3-space in the usual way. The well known rotational symmetry of the monopole's field appears geometrically as its invariance with respect to $F_g \in \mathrm{Aut}\, P$ for $\forall g \in \mathrm{SU}(2)$. This symmetry leads to the conservation of the total angular momentum (Jackiw and Rebbi 1976, Hasenfratz and 't Hooft 1976)

$$J_k = \varepsilon_{kln} x^l p^n + (Q/r) x_k. \tag{3.14}$$

The infinitesimal symmetries resulting in the conservation of Q, Z, p_0, J_k (for any fixed k) commute with each other. Hence we have the Poisson brackets

$$\{Q, Z\} = \{Q, p_0\} = \{Q, J_k\} = \{Z, p_0\} = \{Z, J_k\} = \{p_0, J_k\} = 0, \tag{3.15a}$$

and naturally

$$\{Q, \mathcal{H}\} = \{Z, \mathcal{H}\} = \{p_0, \mathcal{H}\} = \{J_k, \mathcal{H}\} = 0, \tag{3.15b}$$

$$\{J_k, J_l\} = -\varepsilon_{kln} J^n. \tag{3.15c}$$

Here $\mathcal{H}$ is the Hamiltonian (2.5). It is a simple matter to check (3.15) with the aid of the previous formulae (3.2), (3.9), (3.14) and the basic Poisson brackets (2.1).

The spatial trajectory of the particle lies on a rotation cone whose axis is $\boldsymbol{J}$ on account of

$$(x^k/r) J_k = Q. \tag{3.16}$$

Excluding the trivial case of $J^2 = Q^2$ let us introduce the vector $\boldsymbol{R}$ by

$$\boldsymbol{R} = (1 - Q^2/J^2)^{-1/2} [\boldsymbol{r} - (Qr/J^2)\boldsymbol{J}] \tag{3.17}$$

as a new variable instead of $\boldsymbol{r}$ (for fixed $\boldsymbol{J}$, Q and Z). Geometrically, the transformation (3.17) means that one bends down the half-cone determined by (3.16) to the plane perpendicular to its axis $\boldsymbol{J}$. Using this transformation Boulware *et al* (1976) obtained an effective inverse square potential problem for the Dirac monopole. The O(2, 1) 'dynamical' symmetry of the Dirac monopole (Jackiw 1980) is related to an analogous symmetry of the $1/R^2$ potential (de Alfaro *et al* 1976). We note that the Wu–Yang (1968) monopole is an embedded Abelian one and so it inherits the symmetries of the Dirac monopole. In our case (3.10) and (3.17) give rise to a non-relativistic Coulomb problem

$$\mathrm{d}^2\boldsymbol{R}/\mathrm{d}s^2 = Q(Z - Q)\boldsymbol{R}/R^3. \tag{3.18}$$

An explicit solution of (3.10) is found easily using (3.18). The orbital angular momentum and the conserved energy of the central force problem (3.18) give back $\boldsymbol{J}$ and the conserved energy of (3.10), respectively. The energy can be written in the original variables as

$$S = \tfrac{1}{2}[g^{jk} p_j p_k + (p_4)^2 - (Z - Q)^2]. \tag{3.19}$$

According to the well known dynamical symmetry of the Coulomb problem (see, e.g., Bander and Itzykson 1966) the Runge–Lenz vector

$$\boldsymbol{C} = \mathrm{d}\boldsymbol{R}/\mathrm{d}s \times \boldsymbol{J} + Q(Z - Q)\boldsymbol{R}/R \tag{3.20}$$

is an extra conserved quantity for (3.18). It will be convenient to use

$$\boldsymbol{D} = \left(1 - \frac{Q^2}{J^2}\right)^{1/2} \boldsymbol{C} + \frac{Q^2(Z - Q)}{J^2} \boldsymbol{J} \tag{3.21}$$

instead of $\boldsymbol{C}$. In the original variables $\boldsymbol{D}$ is given by the expression

$$D_k = \varepsilon_{kln} p^l J^n + Q(Z - Q)x_k/r. \tag{3.22}$$

$\boldsymbol{D}$ is well defined by (3.22) even if $J^2 = Q^2$. The reader can readily verify that the relations

$$\{D_k, \mathscr{H}\} = \{D_k, Q\} = \{D_k, Z\} = \{D_k, p_0\} = \{D_k, S\} = 0, \tag{3.23a}$$

$$\{J_k, D_l\} = -\varepsilon_{kln} D^n, \tag{3.23b}$$

$$\{D_k, D_l\} = \varepsilon_{kln} J^n(2S), \tag{3.23c}$$

$$\{S, \mathscr{H}\} = \{S, Q\} = \{S, Z\} = \{S, p_0\} = \{S, J_k\} = 0 \tag{3.23d}$$

are valid. Equations (3.23*b*)-(3.23*c*) give us a symmetry algebra which is quite analogous to that of the non-relativistic Coulomb problem (surprisingly enough since our problem is a relativistic one).

In a domain of $E^{\#}$ where $S \neq 0$ let us introduce the quantities

$$H_k = (|2S|)^{-1/2} D_k, \qquad (k = 1, 2, 3) \tag{3.24}$$

instead of D_k. Our main result in this part is that the Poisson brackets

$$\begin{aligned} \{J_k, J_l\} &= -\varepsilon_{kln} J^n, \\ \{J_k, H_l\} &= -\varepsilon_{kln} H^n, \\ \{H_k, H_l\} &= (\operatorname{sgn} S)\varepsilon_{kln} J^n \end{aligned} \tag{3.25}$$

represent a dynamical O(4) or O(3, 1) algebra depending on the sign of the 'effective binding energy' S.

4. The bound state spectrum

In this part of the paper we would like to study the quantum mechanical version of our problem. We shall obtain a bound state spectrum with a high degree of degeneracy as a consequence of the large symmetry algebra described previously. In the derivation of the bound state spectrum we shall follow a train of thought analogous to that used by several authors in the case of the non-relativistic Coulomb problem (see, e.g., Bander and Itzykson 1966).

In an arbitrary quantisation scheme one represents some subset of $C^\infty(E^{\#})$ among the Hermitian operators of a Hilbert space. According to very basic principles of the quantum mechanics the next relation must hold

$$[\hat{f}, \hat{h}] = -\mathrm{i}\hbar\widehat{\{f, h\}}. \tag{4.1}$$

On the left-hand side of (4.1) $[\hat{f}, \hat{h}]$ is the commutator of the Hermitian operators $\hat{f}$, $\hat{h}$ which represent the observables $f, h \in C^\infty(E^{\#})$. From now on we assume that we are given a representation of the basic Poisson brackets (2.1) (at least those brackets not containing x^0 or p_0) by operators $\hat{x}^\mu$, $\hat{p}_\nu$, $\hat{\xi}^a$ which act in a Hilbert space denoted by $\mathscr{H}$. Let us 'define' the operators $\hat{W}$, $\hat{Q}$, $\hat{Z}$, $\hat{J}_k$ $(k = 1, 2, 3)$, $\hat{S}$ with the aid of the expressions (2.4), (3.2), (3.9), (3.14) and (3.19), respectively. We simply substitute the corresponding operators for x^μ, p_ν, ξ^a in the c-number formulae.

Furthermore, let us introduce the operator $\hat{D}_k$ $(k = 1, 2, 3)$ with the help of the formal expression

$$\hat{D}_k = \varepsilon_{kln}(\hat{p}^l\hat{J}^n - \hat{J}^l\hat{p}^n) + \hat{Q}(\hat{Z} - \hat{Q})(\widehat{x_k/r}). \tag{4.2}$$

It is easy to see that the operators introduced in this manner are (at least formally) Hermitian ones.

Later on the following equations will be of great importance

$$\hat{J}_k\hat{D}^k = \hat{D}_k\hat{J}^k = \hat{Q}^2(\hat{Z} - \hat{Q}), \tag{4.3a}$$

$$\hat{D}^2 = 2\hat{S}(\hat{J}^2 - \hat{Q}^2 + \hbar^2) + \hat{Q}^2(\hat{Z} - \hat{Q})^2. \tag{4.3b}$$

Here $\hat{D}^2 = \hat{D}_k\hat{D}^k$ and $\hat{J}^2 = \hat{J}^k\hat{J}_k$. The derivation of these equations is quite straightforward but a bit tedious. One has to do some formal manipulations with the defining formulae of the operators involved in (4.3) utilising the basic commutation relations of $\hat{x}^j$, $\hat{p}_k$, $\hat{\xi}^a$. It should be noted that (4.3b) differs from the corresponding c-number relation in a very essential $\hbar^2$ term.

The isospin operator $\hat{W}$ commutes with the operators of all the observables since W is a Casimir function with respect to $\{\ ,\ \}$. Therefore a superselection rule is valid for the isospin of the test particle. This means that the quantum mechanical state space decomposes as an orthogonal direct sum $\mathcal{H} = \bigoplus_{w\in\mathcal{I}} \mathcal{H}_w$ where $\mathcal{I} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ is the set of possible isospin values in $\hbar$ units. In addition,

$$\hat{W}|w\rangle = \hbar^2 w(w+1)|w\rangle, \qquad \hat{f}|w\rangle \in \mathcal{H}_w \tag{4.4}$$

for all $|w\rangle \in \mathcal{H}_w$, $w \in \mathcal{I}$ and any observable f.

It is convenient to think of $\mathcal{H}$ as being spanned by the orthogonal basis consisting of common (generalised) eigenvectors of the observables

$$\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{J}^2, \hat{J}_3, \hat{D}^2, \hat{D}_3. \tag{4.5}$$

(4.5) is a complete system of commuting observables but not a minimal one. In one respect, the completeness and the minimality of its subsystem $\{\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{J}^2, \hat{J}_3\}$ can be seen from the usual 'minimal coupling realisation' via differential operators. On the other hand, the non-minimality of the system (4.5) follows from the presence of the constraint equations (4.3a)-(4.3b) too.

Denote by $|w, q, z, s\rangle$ a common eigenvector of $\hat{W}$, $\hat{Q}$, $\hat{Z}$, $\hat{S}$ satisfying

$$\begin{aligned}
\hat{W}|w, q, z, s\rangle &= \hbar^2 w(w+1)|w, q, z, s\rangle, \\
\hat{Q}|w, q, z, s\rangle &= \hbar q|w, q, z, s\rangle, \\
\hat{Z}|w, q, z, s\rangle &= \hbar z|w, q, z, s\rangle, \\
\hat{S}|w, q, z, s\rangle &= s|w, q, z, s\rangle.
\end{aligned} \tag{4.6}$$

Furthermore, let $\mathcal{H}_{w,q,z,s}$ be the corresponding common eigensubspace of $\hat{W}$, $\hat{Q}$, $\hat{Z}$, $\hat{S}$. (We allow 'generalised eigenvectors' that can only be normalised in the distribution sense.) Keeping $w \in \mathcal{I}$ fixed, q can take the values $q = -w, -w+1, \ldots, w$ since the electric charge of the test particle is the projection of its isospin onto a direction in its internal space, according to (3.2). In order to see the possible values of z let us remember that $\hat{Z}$ represents the infinitesimal generator of the translations in the extra spacelike direction of $\bar{M}$. Hence the spectrum of $\hat{Z}$ would be $\mathbb{R}$, provided that x^4 varies from $-\infty$ to $+\infty$. In this case $\hat{Z}$ would only have generalised eigenvectors. As far as we are concerned we prefer to consider the extra fifth dimension of the spacetime

$\bar{M}$ as a line compactified to a circle of a 'very small' radius λ, in the spirit of Kaluza and Klein. Keeping in mind this latter possibility z is of the form n_z/λ where n_z is an arbitrary integer. In the usual four-dimensional setting of the test particle's quantum mechanics in external Yang-Mills and Higgs fields ('after' the Yang-Mills → Yang-Mills-Higgs dimensional reduction) only the $z = 0$ sector of $\mathscr{H}$ appears. From a four-dimensional aspect, in our setting too, this sector is distinguished since very large mass terms arise in the sectors with $n_z \neq 0$.

Now we turn to the derivation of the bound state spectrum. Let us remember that a solution of (3.18) (and so of (3.10)) is a bounded one exactly when $S < 0$. That is why a bound state is an eigenvector of $\hat{S}$ which belongs to an eigenvalue s less than zero. In $\mathscr{H}_{w,q,z}$ (common eigensubspace of $\hat{W}$, $\hat{Q}$, $\hat{Z}$) bound states exist if and only if $q(z-q) < 0$. This condition is quite plausible as the effective Coulomb problem (3.18) is attractive supposing that $Q(Z - Q) < 0$. The 'only part' of the above statement follows from the fact that for $q(z-q) \geqslant 0$ the operator $\hat{S}_{|\mathscr{H}_{w,q,z}}$ is positive definite. Let us define the bound state Hilbert space $\mathscr{H}^{\mathrm{B}} \subset \mathscr{H}$ by

$$\mathscr{H}^{\mathrm{B}} = \bigoplus_{w\in\mathscr{I}} \bigoplus_{q=-w}^{w} \bigoplus_{n_z=-\infty}^{+\infty} \bigoplus_{s<0} \mathscr{H}_{w,q,z,s}. \tag{4.7}$$

In (4.7) $q(z-q) < 0$ and our task is to determine the possible negative values of s. In $\mathscr{H}_s$, an eigensubspace of $\hat{S}$ with $s < 0$, we can introduce the operators $\hat{H}_k(s)$ by the formula

$$\hat{H}_k(s) = (-2s)^{-1/2}\hat{D}_{k|\mathscr{H}_s}, \qquad (k = 1,2,3) \tag{4.8a}$$

since $\hat{D}_k(\mathscr{H}_s) \subseteq \mathscr{H}_s$. By means of the orthogonal direct sum decomposition $\mathscr{H}^{\mathrm{B}} = \bigoplus_{s<0} \mathscr{H}_s$ and (4.8a) we define the operators $\hat{H}_k$:

$$\hat{H}_k = \bigoplus_{s<0} \hat{H}_k(s), \qquad (k = 1,2,3). \tag{4.8b}$$

This way we arrive at a representation of the dynamical O(4) algebra (3.25) by $\hat{H}_j$, $\hat{J}_{k|\mathscr{H}^{\mathrm{B}}}$, self-adjoint operators of $\mathscr{H}^{\mathrm{B}}$. Let us introduce the operators

$$\begin{aligned} \hat{K}_j &= \tfrac{1}{2}(\hat{J}_j - \hat{H}_j), \qquad (j = 1,2,3), \\ \hat{N}_k &= \tfrac{1}{2}(\hat{J}_k + \hat{H}_k), \qquad (k = 1,2,3). \end{aligned} \tag{4.9}$$

The system of commuting observables (regarded over $\mathscr{H}^{\mathrm{B}}$)

$$\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{K}^2, \hat{K}_3, \hat{N}^2, \hat{N}_3 \tag{4.10}$$

is more convenient to handle than (4.5), since $\hat{K}_j$, $\hat{N}_k$ give the usual splitting of the O(4) algebra into two independent O(3) factors:

$$\begin{aligned} [\hat{K}_j, \hat{K}_l] &= \mathrm{i}\hbar\varepsilon_{jln}\hat{K}_n, \\ [\hat{N}_j, \hat{N}_l] &= \mathrm{i}\hbar\varepsilon_{jln}\hat{N}_n, \\ [\hat{K}_j, \hat{N}_l] &= 0. \end{aligned} \tag{4.11}$$

For fixed w, q, z, s ($q(z-q) < 0$, $s < 0$) $\mathscr{H}_{w,q,z,s}$ must decompose as an orthogonal direct sum of finite-dimensional Hilbert spaces carrying irreducible, self-adjoint representations of the O(4) algebra (4.11) because $\mathscr{H}_{w,q,z,s}$ is an invariant subspace of this algebra. The unitary equivalence classes of irreducible, self-adjoint representations of the algebra (4.11) can be characterised by the eigenvalues of the Casimir operators $\hat{K}^2$ and $\hat{N}^2$ that belong to the two independent O(3) factors. In general the possible eigenvalues of $\hat{K}^2$ and $\hat{N}^2$ are of the form $\hbar^2 k(k+1)$ and $\hbar^2 n(n+1)$ with arbitrary non-negative half-integers k and n. Now let $|w, q, z, s, k, k_3, n, n_3\rangle \in \mathcal{H}^{\mathrm{B}}$ be a common eigenvector of the system (4.10) with some half-integers k and n. Keeping k and n fixed k_3 and n_3 can run as $k_3 = -k, -k+1, \ldots, k$; $n_3 = -n, -n+1, \ldots, n$, of course. Using the definitions of $\hat{K}_j$, $\hat{N}_k$ we get from (4.3a)-(4.3b) the relations

$$n(n+1) + k(k+1) = \frac{q^2-1}{2} - \frac{\hbar^2 q^2 (z-q)^2}{4s}, \tag{4.12a}$$

$$n(n+1) - k(k+1) = \frac{\hbar q^2 (z-q)}{(-2s)^{1/2}}. \tag{4.12b}$$

The elimination of $(z-q)/(-2s)^{1/2}$ from (4.12) results in the following second-order equation between $k(k+1)$ and $n(n+1)$:

$$2q^2[k(k+1) + n(n+1)] = q^2(q^2-1) + [k(k+1) - n(n+1)]^2. \tag{4.13a}$$

The subsidiary condition

$$\operatorname{sgn}[(n-k)] = \operatorname{sgn}[(z-q)] \tag{4.13b}$$

also holds because of (4.12b). With the notations $n_< = \min(n, k)$, $n_> = \max(n, k)$ the solution of (4.13a) is

$$n_> - n_< = |q|, \qquad (n_< = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots) \tag{4.14a}$$

and according to (4.13b)

$$\begin{aligned} k &= n_<, \qquad n = n_> \qquad \text{if } (z-q) > 0, \\ k &= n_>, \qquad n = n_< \qquad \text{if } (z-q) < 0. \end{aligned} \tag{4.14b}$$

By analogy with the non-relativistic Coulomb problem we introduce a 'principal quantum number' $\mathcal{N}$ with the formula $\mathcal{N} = (2n_< + 1)$. From (4.12) and (4.14) we obtain the main result of this part, namely the discrete spectrum of $\hat{S}$:

$$s = s_{|q|, q(z-q), \mathcal{N}} = -\frac{\hbar^2}{2}\left(\frac{q(z-q)}{\mathcal{N} + |q|}\right)^2, \qquad (\mathcal{N} = 1, 2, \ldots). \tag{4.15}$$

For fixed w, q, z (4.14) and (4.15) provide us with one-to-one correspondences between any two of the following three objects: the eigenvalue s, the principal quantum number $\mathcal{N}$ or the Casimir numbers k and n of the actual representation of the O(4) algebra in $\mathcal{H}_{w,q,z,s}$. The representation of O(4) in $\mathcal{H}_{w,q,z,s}$—characterised by a pair (k, n)—must occur with onefold multiplicity on account of the completeness of the system of commuting observables (4.10). In other words, $\mathcal{H}_{w,q,z,s}$ is an irreducible representation space of the O(4) algebra. From the above it is clear that $\{\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{K}_3, \hat{N}_3\}$ is a (minimal) complete system of observables.

Keeping (w, q, z, s) fixed the pair (k_3, n_3) can have $(2k+1)(2n+1)$ different values where the pair (k, n) describes the actual representation. It is fairly trivial to show that $(2k+1)(2n+1)$, the dimension of $\mathcal{H}_{w,q,z,s}$, is equal to

$$\mathcal{N}(\mathcal{N} + 2|q|). \tag{4.16}$$

The expression (4.15) remains unchanged under the transformation $q \to (-q)$, $z \to (-z)$. This invariance results in an additional twofold degeneracy for a pair of eigenvalues (w, s), beyond $\mathcal{N}(\mathcal{N} + 2|q|)$.

For any $w \in \mathcal{J}$ let Σ_w denote the spectrum of $\hat{S}_{|\mathcal{H}_w}$. From (4.15) it is obvious that $\Sigma_w \subset \Sigma_{(w+v)}$ where v is an arbitrary natural number. Let us remember that $\mathcal{H}_w$ is the state space for a particle of isospin w. So we can express the previous relation by saying: the effective binding energy spectrum of an isoboson (resp. isofermion) is contained in that of any other isoboson (resp. isofermion) with a greater isospin. In the quantum mechanical framework the particles of different total isospin cannot transform into each other.

It is worth mentioning that in the case of fixed (w, q, z, s) the eigenvalue of $\hat{J}^2$—$\hbar^2 j(j+1)$—can vary according to $j = |q|, (|q|+1), \ldots, (\mathcal{N}+|q|-1)$. This can be seen from a transformation between the two orthogonal bases of $\mathcal{H}^{\mathrm{B}}$ corresponding to the following two complete systems of commuting observables: $\{\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{K}_3, \hat{N}_3\}$, $\{\hat{W}, \hat{Q}, \hat{Z}, \hat{S}, \hat{J}^2, \hat{J}_3\}$.

5. Conclusions

Finally we would like to draw the reader's attention to some further questions connected with the subject of this paper. We presented the bound state spectrum for an isospin-carrying test particle in the background field of the '$r_0 = \infty$ monopole'. In this case the scattering phenomena are also highly likely to be calculable on a purely group theoretic basis. The solution presented here could be used as a starting point to perturbative calculations (taking $1/r_0$ as a 'small' parameter) for the test particle's problem in the background field (1.1) with $r_0 \neq \infty$. In such a perturbative investigation one could well use the non-invariance group of the $r_0 = \infty$ problem.

The appearance of the effective Coulomb problem (3.18) suggests that $\mathcal{H}_{w,q,z}$ carries an irreducible unitary representation of the group $O(4, 2)$. It would be interesting to see whether $\mathcal{H}^{\mathrm{B}}$ could be regarded as an irreducible representation space of some group (to be found).

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